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Comparative Similarity in Branching Space-Times

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Abstract My aim in this paper is to investigate the notions of comparative similarity definable in the framework of branching space-times. A notion of this kind is required to give a rigorous Lewis-style semantics of space-time counterfactuals. In turn, the semantical analysis is needed to decide whether the recently proposed proofs of the non-locality of quantum mechanics are correct. From among the three notions of comparative similarity I select two which appear equally good as far as their intuitiveness and algebraic properties are concerned. However, the relations are not transitive, and thus cannot be used in the semantics proposed by Lewis (J. Philos. Log. 2:418–446, 1973), which requires transitivity. Yet they are adequate for the account of Lewis (J. Philos. Log. 10:217–234, 1981).

Keywords Modal logic · Counterfactuals · Comparative similarity · Branching space-times

1 Introduction

Bell-type theorems rely on counterfactual reasoning that refers to space-like separated events. To give an example:

> Suppose that, as a matter of fact, the measurement of $L\alpha$ on the left and the measurement of $R\beta$ on the right were performed, with the result $L\alpha+$ on the left. Then, *if* the same measurement on the left, $L\alpha$, and a different measurement on the right, $R\gamma$, were performed, the result $L\alpha+$ *would* still occur.

Since such counterfactuals refer to space-like separated events, they are called space-time counterfactuals. For the record we note that some space-time counterfactuals of Bell-type theorems refer to probabilities as well, yet this particular variety will not concern us here.

The paper has been read at the Fifth Meeting of IQSA in Cesenatico, was then submitted for publication in this journal and accepted for publication, but . . . it got lost in the production process. It is reproduced here with minor changes only. The paper was a part of a larger project carried out with Thomas Müller, whom I would like to thank for helpful suggestions and comments.

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The issue of the validity of space-time (non-probabilistic) counterfactuals has an important consequence for the foundational debate of quantum mechanics. It has been alleged that, given that some space-time counterfactuals are correct, it can be proved that quantum mechanics is nonlocal—see [11]. One could hope that Lewis's now standard analysis of counterfactuals should resolve the problem. To recall, this analysis postulates a set of possible situations (possible worlds), with the relation of accessibility, and ordered by yet another relation, that of comparative similarity. It further assumes that any sentence of a given language is either true or false in any possible world. Calling a world in which sentence ψ is true a ψ-world, the truth condition for the 'would' counterfactual conditional, $\Box\!\!\rightarrow$, as proposed in [5] is:

> $\psi \Box\!\!\rightarrow \varphi$ is true at the world w iff some (accessible) $\psi \wedge \varphi$-world is closer to w than any $\psi \wedge \neg\varphi$-world, if there are any (accessible) ψ-worlds.

For any world σ, comparative similarity (or comparative closeness, as the quote may suggest) should yield a non-strict ordering $\sqsubseteq_\sigma$, where $\eta \sqsubseteq_\sigma \gamma$ means that γ is not more similar to σ than η, or that η is at least as similar to σ as γ. That is, non-strict and strict relations of comparative similarity are related by: $\eta \sqsubseteq_\sigma \gamma = \neg(\gamma \sqsubset_\sigma \eta)$. Lewis [5] requires that $\sqsubseteq_\sigma$ be a weak ordering, i.e., that $\sqsubseteq_\sigma$ be transitive and connected, where the latter condition is satisfied if for any η and γ: $\eta \sqsubseteq_\sigma \gamma$ or $\gamma \sqsubseteq_\sigma \eta$. The transitivity and connectivity of $\sqsubseteq_\sigma$ are crucial in this standard account of counterfactuals. Given the above truth condition for counterfactuals, a non-transitive frame of possible worlds will render the following form of reasoning invalid, although this form is intuitively correct:

$$\begin{array}{l} \alpha \Box\!\!\rightarrow \beta \\ \alpha \wedge \beta \Box\!\!\rightarrow \gamma \\ \hline \alpha \Box\!\!\rightarrow \gamma \end{array} \tag{1}$$

An example is provided by the non-transitive frame consisting of four possible worlds u, x, y, and z such that $x \sqsubset_u y$, $y \sqsubset_u z$, and $z \sqsubseteq_u x$, where x is an $(\alpha \wedge \beta \wedge \gamma)$-world, y is an $(\alpha \wedge \beta \wedge \neg\gamma)$-world, z is an $(\alpha \wedge \neg\beta \wedge \neg\gamma)$-world, and u is an $(\neg\alpha \wedge \neg\beta \wedge \neg\gamma)$-world.

As it stands, Lewis's analysis can hardly yield a verdict about the validity of space-time counterfactuals because of the vagueness of its concept of comparative similarity. My aim is to stick to Lewis's analysis of counterfactuals, while supplementing it with a precise concept of comparative similarity. I implement Lewis's analysis rather than develop rival theories proposed in the quantum context [3, 12, 13], since I believe that Lewis's account properly captures most of our everyday argumentation involving counterfactuals. In other words, the forms of counterfactual reasoning standardly taken for valid are delivered as valid in Lewis's analysis, and the forms of counterfactual reasoning standardly taken for invalid are delivered as invalid in Lewis's analysis.

To introduce the required notion of comparative similarity, I will use the framework of stochastic outcomes in branching space-times (SOBST), as developed in [4, 9]. The inspiration for these models came from branching space-time of [1] and outcomes in branching time of [2]. In the section that follows I will sketch the SOBST framework, giving it a geometrical twist rather than the usual purely algebraic one. Then, in the next section, I will introduce comparative similarity of histories.

2 A Geometrical Approach to Branching

Our point of departure is the intuition that possibility is a relative concept, as the phrases: 'one event make another event possible' or 'given that one event occurs, some other event is possible' suggest. This at least is the notion of possibility that our quantum speak seems to require. We say, for instance, that $+$ and $-$ are two alternative possible results of a measurement event, meaning that if, in fact, the measurement event occurs, one of two alternative continuations of it, one with the $+$ result or the other with the $-$ result, is to follow. A natural model of a measurement with a few possible continuations consists of two possible histories sharing a common initial segment with the measurement event, but differing in their future parts, as one contains the $+$ result, and the other the $-$ result.[1]

The construction will proceed in two stages. First, we will build a branching structure by pasting together some Minkowski space-times or some space-times of general relativity. Since clearly a collection of space-times falls short of being a representation of possible histories, in the second stage we need to assign states to regions of a branching structure. The resulting object, a branching structure together with states assigned, is to represent a collection of possible histories.

As a way of conveying our intuition, consider how a chancy process taking place at a single point—let us call it, a choice point—and with two possible outcomes, is to be represented in the Minkowski space-times. Consider for example a point-like particle hitting a translucent medium, with two possible outcomes: the particle being transmitted or the particle being reflected. We take two Minkowski space-times, stipulate that a point in one and a point in the other represent the coordinates of our choice point, and then paste the two space-times at these two points and 'below' them, while keeping the space-times 'above' the (pasted) points separate. This brings in a distinction between the choice points, i.e., points active in bringing about chancy events, and surfaces of divergence, which account for the way a chancy process affected at some point(s) propagates globally. It is the shape of the surfaces of divergence that we need to know in order to determine how space-times are pasted.

Now, if two Minkowski space-times split at a single choice point x, two things are intuitively clear: all points in the backward light cone of x are shared by the space-times involved and no point in the forward light cone of x is shared by the two space-times (see the left part of Fig. 1). But what about the 'wings', that is, the totality of points that are neither in the forward nor in the backward light cones of x? Here I fully endorse an argument of Belnap [1, pp. 411–414] to the effect that the wings are shared by the two space-times. To repeat Belnap's argument, suppose that two points y and z, each belonging to a different space-time, are located in the 'wings' of a choice point x, at which a space-time containing x and a space-time containing y split. An 'agent' responsible for y and z being in two alternatives should be located in the backward light cone of y and in the backward light cone of z. For there should be a causal answer to a question like 'although z occurred, why could its alternative y have happened?' However, as far as our story goes, the only agent responsible for the split is the point x, which is outside the backward light cone of y and outside the backward light cone of z. Hence, y and z must be in the shared region of the two space-times.

Before we investigate branching produced by more than a single choice point, let us consider how choice points of two space-times should be located. Indeed, some experiments,

[1] The idea that possible worlds can overlap is characteristic for approaches with branching; in the more popular theory of *divergent worlds* overlapping is strictly forbidden—see [7].

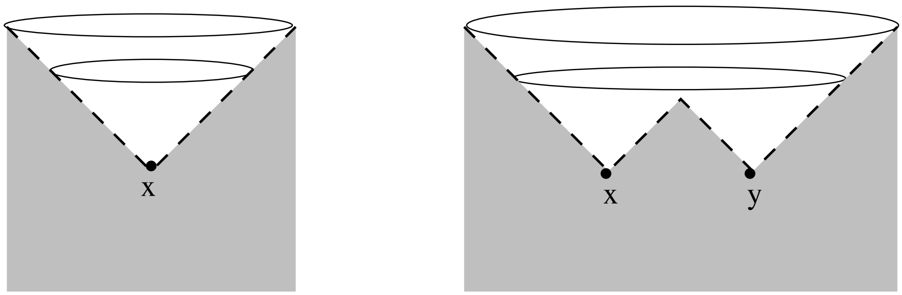

Fig. 1 Space-times separate at a single choice point x (*on the left*), and at two choice points x and y (*on the right*). Regions of overlap are shown *shadowed*. Surfaces of divergence, represented by *broken lines*, are not in the region of overlap

most notably EPR-like experiments, require a few choice points in their models, the choice points being moreover space-like separated. On reflection, this is not accidental: space-like separation is a common feature of choice points of any two branching space-times. For if point x is among the choice points of two space-times σ and γ, then any point in the forward light cone of x is definitely either in σ or in γ, and hence cannot be a choice point of these two space-times. Thus, I will require that no matter how large the set of choice points of two space-times is, any two elements of it must be space-like separated.

As an exercise, let us now consider a surface of divergence of two Minkowski space-times that split at two choice points x and y. In a case like this, depending on the frame of reference, there might be three answers as to where these space-times split: (1) at x, (2) at y, and (3) at x *and* at y. Using Belnap's argument, one arrives at the surface of divergence schematically depicted on the right-hand-side of Fig. 1.

To say, quite generally, what the common segment of two Minkowski space-times is, I first introduce the following ordering:

Definition 1 (Minkowskian Ordering) For four-points x and y from Minkowski space-time σ, we say that $x \leqslant_\sigma y$ iff x lies within or on the backward light cone of y.

It can be easily checked that $\leqslant_\sigma$ is reflexive, anti-symmetric, and transitive, i.e., a partial ordering on space-time σ. It yields a strict partial ordering $<_\sigma$ defined by putting: $x <_\sigma y$ iff $x \leqslant_\sigma y \wedge x \neq y$.

Let us first focus upon only two Minkowski space-times, say, σ_1 and σ_2, whose points are ordered, respectively, by $\leqslant_1$ and $\leqslant_2$, and which split at the choice points forming a non-empty set C_{12}. These space-times should be thought of as two copies of the Minkowski space-time, with points of one space-time being related to points of the other by a 'counterpart relation' R_{12}. The counterpart relation preserves causal orderings, that is, for $x_1 \leqslant_1 y_1$, if $R_{12}(x_1, x_2)$ and $R_{12}(y_1, y_2)$, then $x_2 \leqslant_2 y_2$. Now, for c to be a choice point between σ_1 and σ_2, it must be that $c \in \sigma_1$ and $c \in \sigma_2$. Recall also that choice points of two space-times must be space-like separated. Although the concept of choice point is taken as a primitive, it is convenient to set down these observations as a Fact:

Fact 1 (Set of Choice Points) *For a set $C_{\sigma\eta}$ of choice points for space-times σ and η, any $c \in C_{\sigma\eta}$ is in both σ and η, and any distinct $c_1, c_2 \in C_{\sigma\eta}$ are space-like.*

The common segment of σ and η is produced by this requirement:

$$\text{For } x \in \sigma \text{ and } y \in \eta,\ x = y \text{ iff } R_{\sigma\eta}(x, y) \text{ and } \forall c \in C_{\sigma\eta} \neg(x >_\sigma c) \wedge \neg(y >_\eta c).$$

Here '$=$' stands for identity, so the above condition says that expressions 'x' and 'y' denote the same point that belongs to the two space-times. Note that the second conjunct in this condition is redundant, since given that x and y are counterparts, $x >_\sigma c \Leftrightarrow y >_\eta c$. We further require that no point is shared by σ and η if $C_{\sigma\eta}$ is empty. Note that the points *on* the forward light cone of a choice point are not in the shared region.

We need now to extend this observation to a general definition, which should make clear what the result of pasting a family of Minkowski space-times is. In this object we will have many pairs of Minkowski space-times and thus— many sets of choice points for pairs of space-times. How then do these sets of choice points relate? It suffices to consider three space-times σ, η, and γ and the sets $C_{\sigma\eta}, C_{\sigma\gamma}$, and $C_{\eta\gamma}$ of choice points for corresponding pairs of space-times. First, a point at which σ separated from η is the same as the point at which η separated from σ, hence $C_{\eta\sigma} = C_{\sigma\eta}$. Second, if at the same point σ separates from η and σ separates from γ, then either η and γ are the same, or they separate above or at this point. Third, if σ separated from η at $c_{\sigma\eta}$ and at a later point $c_{\eta\gamma}$ η separated from γ, then σ and γ already separated at $c_{\sigma\eta}$. Finally, if a choice point $c_{\sigma\eta}$ for σ and η is space-like separated from a choice point $c_{\sigma\gamma}$ for σ and γ, then both $c_{\sigma\eta}$ and $c_{\sigma\gamma}$ are choice points for η and γ. Let's put down these observations as a postulate for a proper combination of sets of choice points:[2]

Postulate 1 (Proper Combination of Sets of Choice Points) *For sets $C_{\sigma\eta}, C_{\sigma\gamma}$, and $C_{\eta\gamma}$ of choice points for corresponding pairs of space-times*,

1. $C_{\sigma\eta} = C_{\eta\sigma}$,
2. *if $a \in C_{\sigma\eta}$ and $a \in C_{\sigma\gamma}$, then $C_{\eta\gamma} = \emptyset$ or there is a point $c \in C_{\eta\gamma}$ s.t. $a \leqslant_\eta c$*,
3. *if $a \in C_{\sigma\eta}$, $b \in C_{\sigma\gamma}$, and $a <_\sigma b$, then $a \in C_{\eta\gamma}$*,
4. *if $a \in C_{\sigma\eta}, a \notin C_{\sigma\gamma}$ and $b \in C_{\sigma\gamma}, b \notin C_{\sigma\eta}$, and neither $a \leqslant_\sigma b$ nor $b \leqslant_\sigma a$, then $a, b \in C_{\eta\gamma}$*,
5. *if $C_{\sigma\eta} = \emptyset$, then $C_{\sigma\gamma} = C_{\eta\gamma}$.*

With this postulate, we proceed to define our key concept, i.e., branching structure.

Definition 2 (Branching Structure) Let W' be a family of Minkowski space-times such that any two distinct space-times $\sigma, \eta \in W'$ are related by a counterpart relation $R_{\sigma\eta}$ and have a (possibly empty) set $C_{\sigma\eta}$ of choice points. Let also any three sets $C_{\sigma\eta}, C_{\sigma\gamma}$, and $C_{\eta\gamma}$ of choice points satisfy the postulate of proper combination of sets of choice points. Then W' is a branching structure iff

$$\text{for } x \in \sigma \text{ and } y \in \eta,\ x = y \text{ iff } R_{\sigma\eta}(x, y) \text{ and } \forall c \in C_{\sigma\eta} \neg(x >_\sigma c).$$

The immediate consequence of the counterpart relation is that in any shared region of space-times σ and η, the corresponding orderings coincide:

$$\forall x, y \in \sigma \cap \eta \quad (x \leqslant_\sigma y \Leftrightarrow x \leqslant_\eta y). \tag{2}$$

[2] I am grateful to Thomas Müller for pointing out to me mistakes in an earlier version of this postulate.

This way of pasting space-times has also significant consequences. First, for σ and η from a branching structure, if $x \leqslant_\eta y$ and $y \leqslant_\sigma z$, then $x \leqslant_\sigma z$. Second, if $x \leqslant_\eta y$ and $y \leqslant_\sigma x$, then $x = y$. These two facts allow us to construct a single ordering out of many local orderings—this ordering, call it *full ordering*, is necessary to relate the present geometrical approach to the algebraic framework of stochastic outcomes in branching space-times of [4, 9]. I thus introduce the reflexive, antisymmetric, and transitive ordering $\leqslant$ on the set $W = \{x \mid \exists \langle \sigma, \leqslant_\sigma \rangle \in W' \; x \in \sigma\}$ of all points from a branching structure W':

Definition 3 (Full Ordering) For $x, y \in W$, where W is the set of all points of branching structure W', we say that $x \leqslant y$ iff $\exists \langle \sigma, \leqslant_\sigma \rangle \in W' \;\; x \leqslant_\sigma y$.

By Definitions 2 and 3, it follows that surfaces of divergence are as follows:

Definition 4 (Surface of Divergence) For Minkowski space-times $\langle \sigma, \leqslant_\sigma \rangle$ and $\langle \eta, \leqslant_\eta \rangle$ from a branching structure W', and the set $C_{\sigma\eta}$ of their choice points, z belongs to the surface of divergence $\mathcal{D}(\sigma, \eta)$ iff z lies on the forward light cone of some $c \in C_{\sigma\eta}$ and does not lie within the forward light cone of any $c \in C_{\sigma\eta}$.

Recall that, given our definition of branching structure, a surface of divergence of two space-times does not belong to a region shared by these space-times. Note also that a surface of divergence is constructed out of light cones. This means that relations 'lying on (below or above) a surface of divergence' are Lorentz-invariant. This allows modal statements referring to space and time, like 'Given that event A at (x_1, t_1) occurs, event B at (x_2, t_2) is possible' to have truth-values that are independent from the frame of reference.

It is worth observing that our branching models for the Minkowski space-times can be generalized to other space-times. The above definitions make clear that it is definability, in terms of light cones, of reflexive, antisymmetric, and transitive ordering, that is crucial for this enterprise. Accordingly, instead of referring to the Minkowski space-times, one may straightforwardly postulate that space-times allow for the orderings required. Let us note, however, that for space-times of general relativity, i.e. four dimensional differential manifolds with a Lorentz metric of signature $+2$, there is no uniform way of introducing the antisymmetric orderings that we need here. Some space-times of general relativity, for instance, allow for causal loops, which imply a failure of antisymmetry. What one can do, however, is to delineate a class of space-times of general relativity for which reflexive, antisymmetric, and transitive orderings are definable.

Returning to the construction, since a history is not merely a space-time, a branching structure W' falls short of representing the totality of possible histories. To represent a history, we had better make a complete partition of a space-time into regions, and then assign a state to each region so obtained. By taking this course of action rather than assigning states to points we avoid a philosophically dubious commitment to point-like particulars. Although it is a physical state that we have a clear concept of, we may consider other states as well, i.e., biological, psychical, or whatever we can clearly think of. Moreover, taking a clue from general relativity, the assignment cannot be fully independent from a space-time at hand. To accommodate our causal intuition, as well as to do justice to our physical motivation, the ascription of states to space-times from a given branching structure must be rather subtle. Nevertheless, since this issue is not relevant to the present objective, I do not bring it in here.[3]

[3] I discuss it in detail in [10].

It is enough to remember that histories are obtained from a branching structure by means of assigning states to regions of space-times involved, where these regions completely partition the set of all points of the branching structure. The result of this assignment is a universe of histories, or simply a universe.

Definition 5 (Universe) The pair $\langle W', \mathfrak{A}\rangle$, where W' is a branching structure and $\mathfrak{A}$ is a state assignment on W', is a universe.

This leads to the following definition of a history:

Definition 6 (History) $\langle \sigma, \mathfrak{R}S\rangle$ is a history in universe $\langle W', \mathfrak{A}\rangle$ if $\langle \sigma \leqslant_\sigma\rangle \in W'$ and $\mathfrak{R}S$ is a set of pairs $\langle r, s\rangle$, where r is a region of σ and s is a state assigned by $\mathfrak{A}$ to r, and the set of regions r forms a complete partition of σ.

3 Comparative Similarity

The intuition that underlies the present concept of comparative similarity is that most similar (closest) histories are those that split last. Notably, the same intuition underlies the concept developed in [3]. Recall that a universe of possible histories is supposed to represents possibilities that have been, are, or will be open. We do not need to require, however, that any two histories built upon a branching structure have a non-empty intersection. In a universe there can be a 'loose' possible history, so different from any other that it could not evolve from an initial segment of the latter. Yet, for a history that was, is, or will be possible *with respect to* a given history, it must be that the two histories share an initial segment. Accordingly, we say that:

Definition 7 (Accessibility) For histories $h_1 = \langle \sigma_1, \mathfrak{R}S_1\rangle$ and $h_2 = \langle \sigma_2, \mathfrak{R}S_2\rangle$ in universe $\langle W', \mathfrak{A}\rangle$, we say that h_1 is accessible from h_2 iff $\sigma_1 \cap \sigma_2 \neq \emptyset$.

Clearly accessibility is reflexive, symmetric, and transitive. As required, in a universe that grows from a single trunk, any history is accessible from any other. The immediate consequence of our dictum 'most similar histories split last' is that it is only the geometry of a branching structure that is relevant for the notion of similarity of histories. Thus, in what follows I will often use 'histories' and 'space-times' interchangeably.

To define comparative similarity via the distance of separation of histories we need to appeal to surfaces of divergence. Yet the result is not straightforward, since surfaces of divergence can be ordered in a number of ways, each ordering giving a possibly different verdict about comparative similarity. For three space-times σ, η, and γ, their surfaces of divergence $\mathcal{D}(\sigma, \eta)$, and $\mathcal{D}(\sigma, \gamma)$ can intersect in a rather complicated fashion, depending on the location of the choice points involved.

To begin with the simplest case, if histories σ and η split at a single point c_1 and histories σ and γ also split at a single point c_2, then η is more similar to σ than γ iff $c_1 > c_2$.

Consider next the case in which at least one set $C_{\sigma\eta}$ or $C_{\sigma\gamma}$ of choice points contains more than a single point. Again, if each point of $C_{\sigma\eta}$ is 'above' each point of $C_{\sigma\gamma}$, the decision is straightforward: η is more similar to σ than γ.

There are, however, troublesome cases in which the decision is not easy to make:

$$C_{\sigma\eta} = \{x_1, y_1\}, \qquad C_{\sigma\gamma} = \{x_2, y_2\}, \quad x_1 < x_2 \text{ but } y_1 > y_2, \tag{3}$$

$$C_{\sigma\eta} = \{x_1, y_1\}, \qquad C_{\sigma\gamma} = \{x_2, y_2\}, \quad x_1 < x_2 \text{ but } y_1 = y_2. \tag{4}$$

In both these cases, the observer's answer as to whether η is more similar to σ than γ depends crucially on his location. If he is near x_1 or near x_2, he opts for γ being more similar to σ than η, but if he is near y_1 or near y_2, he believes that η is more similar to σ than γ (the first case), or that η and γ are equally similar to σ (the second case). The discussion suggests the following three candidates for strict comparative similarity $\sqsubset_\sigma$:

Definition 8 (Strong Version of Strict Comparative Similarity) For histories σ, η, and γ, and their sets of choice points $C_{\sigma\eta}$ and $C_{\sigma\gamma}$, respectively, we say that η is more similar to σ than γ, $\eta \sqsubset_\sigma \gamma$, iff $\forall x \in C_{\sigma\gamma} \forall y \in C_{\sigma\eta}\ x < y$.

Definition 9 (Mild Version of Strict Comparative Similarity) For histories σ, η, and γ, and their sets of choice points $C_{\sigma\eta}$ and $C_{\sigma\gamma}$, respectively, we say that η is more similar to σ than γ, $\eta \sqsubset_\sigma \gamma$, iff $\forall x \in C_{\sigma\gamma} \exists y \in C_{\sigma\eta}\ x < y$.

Definition 10 (Weak Version of Strict Comparative Similarity) For histories σ, η, and γ, and their sets of choice points $C_{\sigma\eta}$ and $C_{\sigma\gamma}$, respectively, we say that η is more similar to σ than γ, $\eta \sqsubset_\sigma \gamma$, iff $\forall x \in C_{\sigma\gamma} \exists y \in C_{\sigma\eta}\ x \leqslant y$ and for some $x' \in C_{\sigma\gamma}, y' \in C_{\sigma\eta}\ x' < y'$.

Note that Definition 8 as well as Definition 9 require that surfaces of divergence do not intersect. Under Definition 10, however, η might be more similar to σ than γ, although the corresponding surfaces of divergence $\mathcal{D}(\sigma, \gamma)$ and $\mathcal{D}(\sigma, \eta)$ coincide in some region. Figure 2 compares the verdicts given by the three definitions for pairs of space-times with two choice points. To differentiate between strong comparative similarity and mild comparative similarity, note that Definition 8 is unnecessarily strong. The satisfaction of the clause of Definition 9 suffices for an observer located anywhere on the surface $\mathcal{D}(\sigma, \eta)$ to give a correct verdict that γ is no longer an open possibility. I do not see, however, any reason, intuitive or mathematical, to prefer mild comparative similarity to weak comparative similarity or vice versa.[4] Note that all three relations are transitive, asymmetric, and irreflexive.

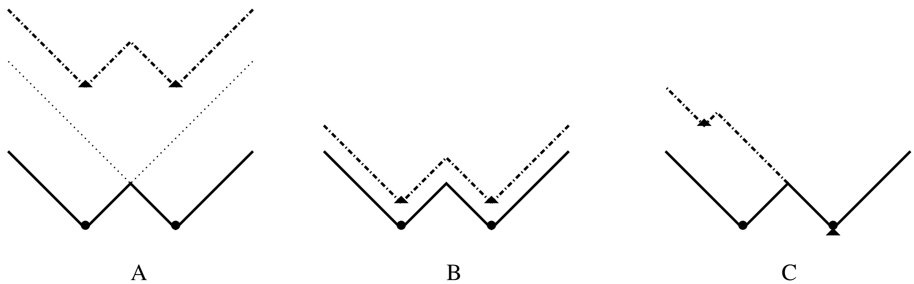

Fig. 2 *Circles* represent choice points of σ and γ, and *triangles* represent choice points of σ and η. By the strong comparative similarity, only in the A scenario η is more similar to σ than γ. By the mild comparative similarity, η is more similar to σ than γ in the A and B scenarios, but not in C scenario. Given the weak comparative similarity, in all the three scenarios η is more similar to σ than γ

[4]For an argument in favor of the weak relation, see [8].

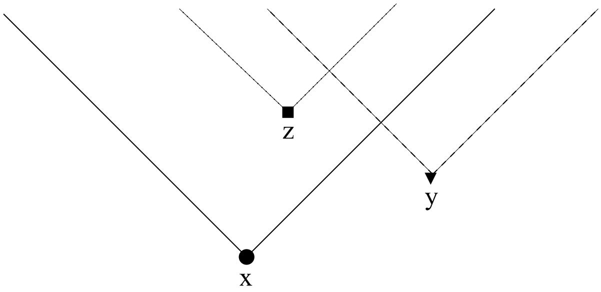

Fig. 3 For $x \in C_{\eta\sigma}$, $y \in C_{\gamma\sigma}$, and $z \in C_{\delta\sigma}$, $\eta \sqsubseteq_\sigma \gamma$ and $\gamma \sqsubseteq_\sigma \delta$ but $\neg\eta \sqsubseteq_\sigma \delta$

For the semantics of Lewis [5], however, we need a non-strict weak ordering. Following Lewis, we may define, for each version of strict comparative similarity, the relation $\sqsubseteq_\sigma$ of non-strict comparative similarity by putting:

$$\eta \sqsubseteq_\sigma \gamma = \neg(\gamma \sqsubset_\sigma \eta).$$

Clearly, with this definition we guarantee connectivity to hold, i.e., we have $\eta \sqsubseteq_\sigma \gamma \vee \gamma \sqsubseteq_\sigma \eta$. Unfortunately, no matter which strict version of comparative similarity we started with, the resulting non-strict relation does not satisfy transitivity. Figure 3 exhibits an arrangement of space-times in which transitivity fails on all three concepts. Is there a remedy for this failure of transitivity?

We have already seen that the combination of the truth condition for counterfactuals of Lewis [5] and the non-transitivity of non-strict comparative similarity leads to incorrect verdicts about the validity of a form of reasoning. Thus, the natural move is to try to appropriately modify the truth condition. In fact, it has already been appropriately modified by Lewis [6].

Before we turn to this modified account, let us reflect on the requirements of connectivity and transitivity of comparative similarity. Both requirements are philosophically questionable, with connectivity being perhaps more troublesome. After all, it requires that any two worlds be comparable with respect to a given third world. Worlds can differ in a huge number of ways, and it appears unreasonable to demand that any two differences can be compared.

Mathematically speaking, we have *two* separate conditions to satisfy, yet the freedom we have in constructing a notion of comparative similarity makes the two tasks merge. Consider the relation $\sim_\sigma$ defined as:

$$\eta \sim_\sigma \gamma \quad \text{iff} \quad \neg(\eta \sqsubset_\sigma \gamma) \wedge \neg(\gamma \sqsubset_\sigma \eta).$$

Can $\sim_\sigma$ be read as a relation of being equally similar to a given world σ? If $\sqsubset_\sigma$ is a partial ordering, then the relata of $\sim_\sigma$ are either (1) identities, or (2) pairs of worlds equally similar to σ, or (3) pairs of incomparable worlds. Can we nevertheless ignore these differences and treat $\sim_\sigma$ as a relation of equal similarity? For this to be possible, since equal similarity should be an equivalence relation, $\sim_\sigma$ must be an equivalence relation as well, i.e., reflexive, symmetric, and transitive. Given the definition of $\sim_\sigma$, it is only transitivity that can be problematic. Transitivity is satisfied, however, if $\neg(\sqsubset_\sigma)$ is transitive, in which case there is no obstacle to consider $\sim_\sigma$ a relation of equal comparative similarity.

Our problem, however, is that on each of the three versions of strict comparative similarity $\sqsubset_\sigma$ that we defined, neither $\neg(\sqsubset_\sigma)$ nor $\sim_\sigma$ is transitive. Yet, each version of our strict comparative similarity is a transitive and antisymmetric relation, i.e. a strict partial ordering.

Thus, by putting:

$$\eta \sqsubseteq_\sigma \gamma \Leftrightarrow \eta \sqsubset_\sigma \gamma \vee \eta = \gamma$$

we obtain a partial ordering. It is precisely this case, non-strict comparative similarity forming a partial ordering and non-transitive $\sim_\sigma$, to which the truth condition for counterfactuals of Lewis [6] applies:

> $\psi \Box\!\!\rightarrow \varphi$ is true at the world w iff for any ψ-world h, there is some ψ-world j such that (i) $j \sqsubseteq_w h$, and (ii) φ holds at any ψ-world k such that $k \sqsubseteq_w j$.

4 Conclusions

Our enterprise turned out to be only half-successful; there are two reasons for this. First, we have two apparently equally good notions of strict comparative similarity: mild and week. Second, each of these strict relations yields a version of non-strict comparative similarity that is non-transitive. This means that none of the three versions of non-strict comparative similarity can be used together with the truth condition for counterfactuals of Lewis [5]. However, since every version of strict comparative similarity that we defined is a strict partial ordering, it can be used in the analysis of counterfactuals of Lewis [6].

References

1. Belnap, N.: Branching space-time. Synthese **92**, 385–434 (1992)
2. Belnap, N.: Various notes on outcomes in branching histories. Unpublished manuscript, Pittsburgh University (1995)
3. Finkelstein, J.: Space-time counterfactuals. Synthese **119**(3), 287–298 (1999)
4. Kowalski, T., Placek, T.: Outcomes in branching space-time and GHZ-Bell theorems. Br. J. Philos. Sci. **50**, 349–375 (1999)
5. Lewis, D.: Counterfactuals and comparative possibility. J. Philos. Log. **2**, 418–446 (1973). Reprinted in Lewis, D.: Philosophical Papers, vol.II. Oxford University Press, Oxford (1986)
6. Lewis, D.: Ordering semantics and premise semantics for counterfactuals. J. Philos. Log. **10**, 217–234 (1981)
7. Lewis, D.: New work for a theory of universals. Australas. J. Philos. **61**, 343–377 (1983)
8. Müller, T.: Branching space-time, modal logic, and the counterfactual conditional. In: Placek, T., Butterfield, J. (eds.) Non-locality and Modality. NATO Science Series. Kluwer Academic, Dordrecht (2002)
9. Placek, T.: Stochastic outcomes in branching space-time. An analysis of the Bell theorems. Br. J. Philos. Sci. **51**(3), 445–475 (2000)
10. Placek, T.: Indeterminism is enough: a branching analysis of Bell-type theorems. In: Placek, T., Butterfield, J. (eds.) Non-locality and Modality. NATO Science Series. Kluwer Academic, Dordrecht (2002)
11. Stapp, H.P.: Nonlocal character of quantum theory. Am. J. Phys. **65**(4), 300–304 (1997)
12. Stapp, H.P.: (2000). Nonlocality, counterfactuals, and consistent histories. quant-ph/9905055
13. Vaidman, L.: Time symmetrized counterfactuals in quantum theory. Found. Phys. **29**(5), 755–765 (1999)